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Green's function for the relativistic Coulomb system via sum over perturbation series

De-Hone Lin†

Department of Physics, National Tsing Hua University, Hsinchu 30043, Taiwan, Republic of China

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Abstract. We evaluate the Green's function of the D-dimensional relativistic Coulomb system via sum over perturbation series which is obtained by expanding the exponential containing the potential term $V(\boldsymbol{x})$ in the path integral into a power series. The energy spectra and wavefunctions are extracted from the resulting amplitude.

1. Introduction

Most physical problems cannot be solved exactly. It is therefore necessary to develop approximation procedures which allow us to approach the exact result with appropriate accuracy. An important approximation method for solving problems in quantum mechanics (QM) is the Rayleigh–Schrödinger perturbation theory. It provides us an effective method of calculating approximate solutions to many problems which cannot be exactly solved by using the Schrödinger equation. Similar to the standard QM, the perturbation method can be developed in the path integral framework of QM [1]. Historically of utmost importance was the application of the perturbation expansion of path integral to the quantum electrodynamics by Feynman [2], from which he first derived 'Feynman's rules', which provide an extremely effective method to calculate the perturbation series and a clear, neat interpretation of the interaction picture.

In the past 10 years, perturbation expansion of the path integral has been used to obtain the exact Green's functions for δ-function potential problems [3–5, 7], non-relativistic Coulomb system [6], and to yield the Dirichlet boundary condition by summing the δ-function perturbation series [8, 9].

In this paper, we would like to add a further application of the perturbation method of the path integral. We calculate the Green's function of a D-dimensional relativistic Coulomb system via summing over the perturbation series. The energy spectra and wavefunctions are extracted from the resulting amplitude.

2. Path integral for the relativistic Coulomb system via sum over the perturbation series

Let us first consider a point particle of mass M moving at a relativistic velocity in a $(D+1)$-dimensional Minkowski space with a given electromagnetic field. By using

† E-mail address: d793314@phys.nthu.edu.tw

$t = -\mathrm{i}\tau = -\mathrm{i}x^4/c$, the path integral representation of the Green's function is conveniently formulated in a $(D+1)$-Euclidean spacetime with the Euclidean metric,

$$(g_{\mu\nu}) = \mathrm{diag}\,(1, \ldots, 1, c^2) \tag{1}$$

and it is given by [10, 11]

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2Mc} \int_0^\infty \mathrm{d}S \int D\rho\Phi[\rho] \int D^D x\, \mathrm{e}^{-A_E/\hbar}. \tag{2}$$

The action integral

$$A_E = \int_{\tau_a}^{\tau_b} \mathrm{d}\tau \left[\frac{M}{2\rho(\tau)} \boldsymbol{x}'^2(\tau) - \mathrm{i}\frac{e}{c}\boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{x}'(\tau) - \rho(\tau)\frac{(E - V(\boldsymbol{x}))^2}{2Mc^2} + \rho(\tau)\frac{Mc^2}{2} \right] \tag{3}$$

where S is defined by

$$S = \int_{\tau_a}^{\tau_b} \mathrm{d}\tau\, \rho(\tau) \tag{4}$$

in which $\rho(\tau)$ is an arbitrary dimensionless fluctuating scale variable, and $\Phi[\rho]$ is some convenient gauge-fixing functional, such as $\Phi[\rho] = \delta[\rho - 1]$, to fix the value of $\rho(\tau)$ to unity [10, 11]. $\hbar/Mc$ is the well known Compton wavelength of a particle of mass M, $\boldsymbol{A}(\boldsymbol{x})$ is the vector potential, $V(\boldsymbol{x})$ is the scalar potential, E is the system energy, and $\boldsymbol{x}$ is the spatial part of the $(D+1)$ vector $x = (\boldsymbol{x}, \tau)$. This path integral forms the basis for studying relativistic potential problems.

Expanding the potential term $V(\boldsymbol{x})$ into a power series and interchanging the order of integration and summation, we obtain the result

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2Mc} \int_0^\infty \mathrm{d}S \int D\rho\Phi[\rho] \mathrm{e}^{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} \mathrm{d}\tau\rho(\tau)\mathcal{E}} K(\boldsymbol{x}_b, \boldsymbol{x}_a; \tau_b - \tau_a) \tag{5}$$

with the series expansion of the pseudotime propagator

$$\begin{aligned} K(\boldsymbol{x}_b, \boldsymbol{x}_a; \tau_b - \tau_a) = \Bigg\{ & K_0 + \sum_{n=1}^\infty \frac{1}{n!}\left(-\frac{\beta}{\hbar}\right)^n \\ & \times \int D^D x \mathrm{e}^{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} \mathrm{d}\tau[\frac{M}{2\rho(\tau)}\boldsymbol{x}'^2(\tau) - \mathrm{i}\frac{e}{c}\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{x}'(\tau) - \rho(\tau)\frac{V(\boldsymbol{x})^2}{2Mc^2}]} \int_{\tau_a}^{\tau_b} \mathrm{d}\tau_1 \rho(\tau_1) V(\boldsymbol{x}(\tau_1)) \\ & \times \int_{\tau_a}^{\tau_b} \mathrm{d}\tau_2 \rho(\tau_2) V(\boldsymbol{x}(\tau_2)) \cdots \int_{\tau_a}^{\tau_b} \mathrm{d}\tau_n \rho(\tau_n) V(\boldsymbol{x}(\tau_n)) \Bigg\} \end{aligned} \tag{6}$$

where we have defined the quantities $\beta = E/Mc^2$, $\mathcal{E} = (M^2c^4 - E^2)/2Mc^2$, and

$$K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \tau_b - \tau_a) = \int D^D x \mathrm{e}^{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} \mathrm{d}\tau[\frac{M}{2\rho(\tau)}\boldsymbol{x}'^2(\tau) - \mathrm{i}\frac{e}{c}\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{x}'(\tau) - \rho(\tau)\frac{V(\boldsymbol{x})^2}{2Mc^2}]}. \tag{7}$$

Ordering the τ as $\tau_1 < \tau_2 < \cdots < \tau_n < \tau_b$ and denoting $\boldsymbol{x}(\tau_k) = \boldsymbol{x}_k$, the perturbative series in equation (6) turns into [1]

$$\begin{aligned} K(\boldsymbol{x}_b, \boldsymbol{x}_a; \tau_b - \tau_a) = {} & K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \tau_b - \tau_a) + \sum_{n=1}^\infty \left(-\frac{\beta}{\hbar}\right)^n \int_{\tau_a}^{\tau_b} \mathrm{d}\tau_n \int_{\tau_a}^{\tau_n} \mathrm{d}\tau_{n-1} \ldots \int_{\tau_a}^{\tau_2} \mathrm{d}\tau_1 \\ & \times \int \left[\prod_{j=0}^n K_0(\boldsymbol{x}_{j+1}, \boldsymbol{x}_j; \tau_{j+1} - \tau_j) \right] \prod_{k=1}^n \rho_k V(\boldsymbol{x}_k)\, \mathrm{d}\boldsymbol{x}_k \end{aligned} \tag{8}$$

where $\tau_0 = \tau_a$, $\tau_{n+1} = \tau_b$, $\boldsymbol{x}_{n+1} = \boldsymbol{x}_b$, and $\boldsymbol{x}_0 = \boldsymbol{x}_a$. In the case of an attractive Coulomb potential, we have

$$\boldsymbol{A}(\boldsymbol{x}) = 0 \qquad V(r) = -\frac{e^2}{r}. \tag{9}$$

The perturbative expansion in equation (8) becomes

$$K(\boldsymbol{x}_b, \boldsymbol{x}_a; \tau_b - \tau_a) = K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \tau_b - \tau_a) + \sum_{n=1}^{\infty} \left(\frac{\beta \mathrm{e}^2}{\hbar}\right)^n \int_{\tau_a}^{\tau_b} \mathrm{d}\tau_n \int_{\tau_a}^{\tau_n} \mathrm{d}\tau_{n-1} \ldots \int_{\tau_a}^{\tau_2} \mathrm{d}\tau_1$$

$$\times \int \left[\prod_{j=0}^{n} K_0(\boldsymbol{x}_{j+1}, \boldsymbol{x}_j; \tau_{j+1} - \tau_j)\right] \prod_{k=1}^{n} \rho_k \frac{\mathrm{d}\boldsymbol{x}_k}{r_k}. \tag{10}$$

The corresponding amplitude K_0 takes the form

$$K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \tau_b - \tau_a) = \int D^D x \mathrm{e}^{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} \mathrm{d}\tau \left[\frac{M}{2\rho(\tau)} \boldsymbol{x}'^2(\tau) - \rho(\tau)\frac{\hbar^2}{2M}\frac{\alpha^2}{r^2}\right]} \tag{11}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant. We now choose $\Phi[\rho] = \delta[\rho - 1]$ to fix the value of $\rho(\tau)$ to unity. The Green's function in equation (5) becomes

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2Mc} \int_0^{\infty} \mathrm{d}S \mathrm{e}^{-\frac{\mathcal{E}}{\hbar}S} \Bigg\{ K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; S) + \sum_{n=1}^{\infty} \left(\frac{\beta e^2}{\hbar}\right)^n \int_{\tau_a}^{\tau_b} \mathrm{d}\tau_n \int_{\tau_a}^{\tau_n} \mathrm{d}\tau_{n-1}$$

$$\ldots \int_{\tau_a}^{\tau_2} \mathrm{d}\tau_1 \int \left[\prod_{j=0}^{n} K_0(\boldsymbol{x}_{j+1}, \boldsymbol{x}_j; \tau_{j+1} - \tau_j)\right] \prod_{k=1}^{n} \frac{\mathrm{d}\boldsymbol{x}_k}{r_k} \Bigg\}. \tag{12}$$

We observe that the integration over S is a Laplace transformation. Because of the convolution property of the Laplace transformation, we obtain

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2Mc} \left\{ G_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \mathcal{E}) + \sum_{n=1}^{\infty} \left(\frac{\beta \mathrm{e}^2}{\hbar}\right)^n \int \left[\prod_{j=0}^{n} G_0(\boldsymbol{x}_{j+1}, \boldsymbol{x}_j; \mathcal{E})\right] \prod_{k=1}^{n} \frac{\mathrm{d}\boldsymbol{x}_k}{r_k} \right\}. \tag{13}$$

We now perform the angular decomposition of equation (13) [11–13]. This can be reached by inserting in equation (13) the expansion of G_0 in term of the D-dimensional hyperspherical harmonics $Y_{l\boldsymbol{m}}(\hat{\boldsymbol{x}})$ [14]:

$$G_0(\boldsymbol{x}_{j+1}, \boldsymbol{x}_j; \mathcal{E}) = \frac{M}{\hbar (r_{j+1} r_j)^{D/2-1}} \sum_{l=0}^{\infty} g_l^0(r_{j+1}, r_j; \mathcal{E}) \sum_{\boldsymbol{m}} Y_{l\boldsymbol{m}}(\hat{\boldsymbol{x}}_{j+1}) Y_{l\boldsymbol{m}}^*(\hat{\boldsymbol{x}}_j) \tag{14}$$

where the g_l^0 is given by [13]

$$\int_0^{\infty} \frac{\mathrm{d}S}{S} \mathrm{e}^{-\frac{\mathcal{E}}{\hbar}S} \mathrm{e}^{-M(r_{j+1}^2 + r_j^2)/2\hbar S} I_{\sqrt{(l+D/2-1)^2 - \alpha^2}} \left(\frac{M}{\hbar}\frac{r_{j+1} r_j}{S}\right). \tag{15}$$

The notation I denotes the modified Bessel function. Integrating over the intermediate angular part of equation (13), we arrive at

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2Mc} \sum_{l=0}^{\infty} G_l(r_b, r_a; \mathcal{E}) \sum_{\boldsymbol{m}} Y_{l\boldsymbol{m}}(\hat{\boldsymbol{x}}_b) Y_{l\boldsymbol{m}}^*(\hat{\boldsymbol{x}}_a). \tag{16}$$

The pure radial amplitude $G_l(r_b, r_a; \mathcal{E})$ has the form

$$G_l(r_b, r_a; \mathcal{E}) = \frac{M}{\hbar} \frac{1}{(r_b r_a)^{D/2-1}} \sum_{n=0}^{\infty} \left(\frac{M\beta e^2}{\hbar^2}\right)^n g_l^{(n)}(r_b, r_a; \mathcal{E}) \tag{17}$$

with $g_l^{(n)}$ given by

$$g_l^{(n)}(r_b, r_a; \mathcal{E}) = \int_0^{\infty} \cdots \int_0^{\infty} \left[\prod_{j=0}^{n} g_l^{(0)}(r_{j+1}, r_j; \mathcal{E})\right] \prod_{k=1}^{n} \mathrm{d}r_k. \tag{18}$$

To obtain the explicit result of $g_l^{(n)}$, we note that

$$\int_0^\infty \frac{\mathrm{d}S}{S}\mathrm{e}^{-\frac{\mathcal{E}}{\hbar}S}\mathrm{e}^{-M(r_b^2+r_a^2)/2\hbar S} I_{\sqrt{(l+D/2-1)^2-\alpha^2}}\left(\frac{M}{\hbar}\frac{r_b r_a}{S}\right)$$
$$= 2\int_0^\infty \mathrm{d}z\frac{1}{\sinh z}\mathrm{e}^{-\kappa(r_b+r_a)\coth z} I_{2\sqrt{(l+D/2-1)^2-\alpha^2}}\left(\frac{2\kappa\sqrt{r_b r_a}}{\sinh z}\right) \tag{19}$$

with $\kappa = \sqrt{M^2c^4 - E^2}/\hbar c$. The equality in equation (19) can be easily proved by the formulae

$$\int_0^\infty \mathrm{d}y\,\frac{\mathrm{e}^{2\nu y}}{\sinh y}\exp\left[-\frac{t}{2}(\zeta_a+\zeta_b)\coth y\right] I_\mu\left(\frac{t\sqrt{\zeta_b\zeta_a}}{\sinh y}\right)$$
$$= \frac{\Gamma((1+\mu)/2-\nu)}{t\sqrt{\zeta_b\zeta_a}\Gamma(\mu+1)} W_{\nu,\mu/2}(t\zeta_b)M_{\nu,\mu/2}(t\zeta_a) \tag{20}$$

with the range of validity

$$\zeta_b > \zeta_a > 0$$
$$\mathrm{Re}\,[(1+\mu)/2-\nu] > 0$$
$$\mathrm{Re}\,(t) > 0,\ |\arg t| < \pi$$

where $M_{\mu,\nu}$ and $W_{\mu,\nu}$ are the Whittaker functions, and

$$\int_0^\infty \frac{\mathrm{d}y}{y}\mathrm{e}^{-zy}\mathrm{e}^{-(a^2+b^2)/y} I_\nu\left(\frac{2ab}{y}\right) = 2I_\nu(2a\sqrt{z})K_\nu(2b\sqrt{z}) \tag{21}$$

with the range of validity

$$a < b \qquad \mathrm{Re}\,z > 0.$$

From equation (19), using the formula

$$\int_0^\infty \mathrm{d}r\, r\,\mathrm{e}^{-r^2/a} I_\nu(\varsigma r)I_\nu(\xi r) = \frac{a}{2}\mathrm{e}^{a(\xi^2+\varsigma^2)/4} I_\nu\left(\frac{a\xi\varsigma}{2}\right) \tag{22}$$

we obtain

$$g_l^{(1)}(r_b, r_a; \mathcal{E}) = \int_0^\infty g_l^{(0)}(r_b, r; \mathcal{E}) g_l^{(0)}(r, r_a; \mathcal{E})\,\mathrm{d}r$$
$$= \frac{2^2}{\kappa}\int_0^\infty z h(z)\,\mathrm{d}z \tag{23}$$

where the function $h(z)$ is defined as

$$h(z) = \frac{1}{\sinh z}\mathrm{e}^{-\kappa(r_b+r_a)\coth z} I_{2\sqrt{(l+D/2-1)^2-\alpha^2}}\left(\frac{2\kappa\sqrt{r_b r_a}}{\sinh z}\right). \tag{24}$$

The expression for $g_l^{(n)}(r_b, r_a; \mathcal{E})$ can be obtained by induction with respect to n, and is given by

$$g_l^{(n)}(r_b, r_a; \mathcal{E}) = \frac{2^{n+1}}{n!}\frac{1}{\kappa^n}\int_0^\infty z^n h(z)\,\mathrm{d}z. \tag{25}$$

Inserting the expression in equation (17), we obtain

$$G_l(r_b, r_a; \mathcal{E}) = \frac{M}{\hbar}\frac{2}{(r_b r_a)^{D/2-1}}$$
$$\times \int_0^\infty \mathrm{d}z\,\mathrm{e}^{\left(\frac{2M\beta e^2}{\hbar^2\kappa}\right)z}\frac{1}{\sinh z}\mathrm{e}^{-\kappa(r_b+r_a)\coth z} I_{2\sqrt{(l+D/2-1)^2-\alpha^2}}\left(\frac{2\kappa\sqrt{r_b r_a}}{\sinh z}\right). \tag{26}$$

With help of the formula in equation (20), we complete the integration of equation (26), and find the radial Green's function for $r_b > r_a$ in the closed form,

$$
\begin{aligned}
G_l(r_b, r_a; E) = {} & \frac{1}{(r_b r_a)^{(D-1)/2}} \frac{Mc}{\sqrt{M^2c^4 - E^2}} \\
& \times \frac{\Gamma\left(1/2 + \sqrt{(l + D/2 - 1)^2 - \alpha^2} - \frac{E\alpha}{\sqrt{M^2c^4 - E^2}}\right)}{\Gamma\left(1 + 2\sqrt{(l + D/2 - 1)^2 - \alpha^2}\right)} \\
& \times W_{\frac{E\alpha}{\sqrt{M^2c^4-E^2}}, \sqrt{(l+D/2-1)^2-\alpha^2}}\left(\frac{2}{\hbar c}\sqrt{M^2c^4 - E^2}\, r_b\right) \\
& \times M_{\frac{E\alpha}{\sqrt{M^2c^4-E^2}}, \sqrt{(l+D/2-1)^2-\alpha^2}}\left(\frac{2}{\hbar c}\sqrt{M^2c^4 - E^2}\, r_a\right).
\end{aligned}
\tag{27}
$$

The energy spectra and wavefunctions can be extracted from the poles of equation (27). For convenience, we define the following variables

$$
\begin{aligned}
\kappa &= \frac{1}{\hbar c}\sqrt{M^2c^4 - E^2} \\
\nu &= \frac{\alpha E}{\sqrt{M^2c^4 - E^2}} \\
\tilde{l} &= \sqrt{(l + D/2 - 1)^2 - \alpha^2} - \frac{1}{2}.
\end{aligned}
\tag{28}
$$

From the poles of $G_l(r_b, r_a; E)$ we find that the energy levels must satisfy the equality

$$
-\nu + \tilde{l} + 1 = -n_r \qquad n_r = 0, 1, 2, 3, \ldots. \tag{29}
$$

Expanding this equation into powers of α, we get

$$
\begin{aligned}
E_{nl} \approx \pm Mc^2 \Bigg\{ & 1 - \frac{1}{2}\left[\frac{\alpha}{n + \frac{1}{2}(D-3)}\right]^2 - \frac{\alpha^4}{[n + \frac{1}{2}(D-3)]^3} \\
& \times \left[\frac{1}{2[l + \frac{1}{2}(D-2)]} - \frac{3}{8}\frac{1}{[n + \frac{1}{2}(D-3)]}\right] + \mathrm{O}(\alpha^6)\Bigg\}.
\end{aligned}
\tag{30}
$$

Here n is defined by $n_r = n - l - 1$. We point out that by setting $D = 3$, the energy levels reduce to the well known form

$$
E_{nl} \approx \pm Mc^2 \left\{1 - \frac{1}{2}\left(\frac{\alpha}{n}\right)^2 - \frac{\alpha^4}{n^3}\left[\frac{1}{2l+1} - \frac{3}{8n}\right] + \mathrm{O}(\alpha^6)\right\}. \tag{31}
$$

The pole positions, which satisfy $\nu = \tilde{n}_l \equiv n + \tilde{l} - l$ $(n = l+1, l+2, l+3, \ldots)$, correspond to the bound states of the D-dimensional relativistic Coulomb system. Near the positive-energy poles, we use the behaviour for $\nu \approx \tilde{n}_l$,

$$
-\Gamma(-\nu + \tilde{l} + 1)\frac{M}{\hbar\kappa} \approx \frac{(-)^{n_r}}{\tilde{n}_l^2 n_r!}\frac{1}{\tilde{a}_H}\left(\frac{E}{Mc^2}\right)^2 \frac{2\hbar Mc^2}{E^2 - E_{nl}^2} \tag{32}
$$

with $\tilde{a}_H \equiv a_H \frac{Mc^2}{E}$ being the modified energy-dependent Bohr radius and $n_r = n - l - 1$ the radial quantum number, to extract the wavefunctions of the D-dimensional Coulomb system

$$
G_l(r_b, r_a; E) = -\frac{\mathrm{i}}{(r_b r_a)^{(D-1)/2}} \sum_{n=l+1}^{\infty} \left(\frac{E}{Mc^2}\right)^2 \frac{2\hbar Mc^2}{E^2 - E_{nl}^2}
$$

$$\times \frac{1}{[(2\tilde{l}+1)!]^2} \frac{1}{\tilde{n}_l^2 \tilde{a}_H} \frac{(\tilde{n}_l+\tilde{l})!}{(n-l-1)!} \mathrm{e}^{-(r_b+r_a)/\tilde{a}_H \tilde{n}_l} \left(\frac{2r_b}{\tilde{a}_H \tilde{n}_l} \frac{2r_a}{\tilde{a}_H \tilde{n}_l} \right)^{\tilde{l}+1}$$
$$\times M\left(-n+l+1, 2\tilde{l}+2; \frac{2r_b}{\tilde{a}_H \tilde{n}_l}\right) M\left(-n+l+1, 2\tilde{l}+2; \frac{2r_a}{\tilde{a}_H \tilde{n}_l}\right)$$
$$= -\frac{\mathrm{i}}{(r_b r_a)^{(D-1)/2}} \sum_{n=l+1}^{\infty} \left(\frac{E}{Mc^2}\right)^2 \frac{2\hbar Mc^2}{E^2 - E_{nl}^2} R_{nl}(r_b) R_{nl}^*(r_a) + \cdots \tag{33}$$

where we have expressed the Whittaker function $M_{\lambda,\mu}(z)$ in terms of the Kummer functions $M(a, b; z)$,

$$M_{\lambda,\mu}(z) = z^{\mu+1/2} \mathrm{e}^{-z/2} M(\mu - \lambda + \tfrac{1}{2}, 2\mu + 1; z). \tag{34}$$

From this we obtain the radial wavefunctions

$$R_{nl}(r) = \frac{1}{\tilde{n}_l \tilde{a}_H^{1/2}} \frac{1}{(2\tilde{l}+1)!} \sqrt{\frac{(\tilde{n}_l+\tilde{l})!}{(n-l-1)!}} \left(\frac{2r}{\tilde{a}_H \tilde{n}_l}\right)^{\tilde{l}+1}$$
$$\times \mathrm{e}^{-r/\tilde{a}_H \tilde{n}_l} M\left(-n+l+1, 2\tilde{l}+2; \frac{2r}{\tilde{a}_H \tilde{n}_l}\right). \tag{35}$$

The normalized wavefunctions are given by

$$\Psi_{nlm}(\boldsymbol{x}) = \frac{1}{r^{(D-1)/2}} R_{nl}(r) Y_{lm}(\hat{\boldsymbol{x}}). \tag{36}$$

Before extracting the continuous wavefunction we note that the parameter κ is real for $|E| < Mc^2$. For $|E| > Mc^2$, the square root in equation (28) has two imaginary solutions

$$\kappa = \mp \mathrm{i}\tilde{k} \qquad \tilde{k} = \frac{1}{\hbar c} \sqrt{E^2 - M^2 c^4} \tag{37}$$

corresponding to

$$\nu = \pm \mathrm{i}\tilde{\nu} \qquad \tilde{\nu} = \frac{E\alpha}{\hbar c \tilde{k}}. \tag{38}$$

Therefore the amplitude has a right-handed cut for $E > Mc^2$ and $E < -Mc^2$. For simplicity, we will only consider the positive energy cut.

The continuous wavefunction is recovered from the discontinuity of the amplitudes $G_l(r_b, r_a; E)$ across the cut in the complex E plane. Hence we have

$$\mathrm{disc}\, G_l(r_b, r_a; E > Mc^2) = G_l(r_b, r_a; E + \mathrm{i}\eta) - G_l(r_b, r_a; E - \mathrm{i}\eta) = -\frac{\mathrm{i}}{(r_b r_a)^{(D-1)/2}}$$
$$\times \frac{M}{\hbar \tilde{k}} \left[\frac{\Gamma(-\mathrm{i}\tilde{\nu} + \tilde{l} + 1)}{(2\tilde{l}+1)!} W_{\mathrm{i}\tilde{\nu}, \tilde{l}+1/2}(-2\mathrm{i}\tilde{k} r_b) M_{\mathrm{i}\tilde{\nu}, \tilde{l}+1/2}(-2\mathrm{i}\tilde{k} r_a) + (\tilde{\nu} \to -\tilde{\nu}) \right]. \tag{39}$$

Using the relations

$$M_{\kappa,\mu}(z) = \mathrm{e}^{\pm \mathrm{i}\pi(2\mu+1)/2} M_{-\kappa,\mu}(-z) \tag{40}$$

where the sign is positive or negative depending on whether $\mathrm{Im}\, z > 0$ or $\mathrm{Im}\, z < 0$, and

$$W_{\lambda,\mu}(z) = \mathrm{e}^{\mathrm{i}\pi\lambda} \mathrm{e}^{-\mathrm{i}\pi(\mu+\frac{1}{2})} \frac{\Gamma(\mu+\lambda+\frac{1}{2})}{\Gamma(2\mu+1)} \left[M_{\lambda,\mu}(z) - \frac{\Gamma(2\mu+1)}{\Gamma(\mu-\lambda+\frac{1}{2})} \mathrm{e}^{-\mathrm{i}\pi\lambda} W_{-\lambda,\mu}(\mathrm{e}^{-\mathrm{i}\pi} z) \right] \tag{41}$$

which is valid only for arg $(z) \in (-\pi/2, 3\pi/2)$ and $2\mu \neq -1, -2, -3, \ldots$. The discontinuity of the amplitude is found to be

$$\begin{aligned}\mathrm{disc}\, G_l(r_b, r_a; E > Mc^2) = &-\frac{\mathrm{i}}{(r_b r_a)^{(D-1)/2}} \frac{M}{\hbar\tilde{k}} \frac{|\Gamma(-\mathrm{i}\tilde{\nu} + \tilde{l} + 1)|^2}{|\Gamma(2\tilde{l} + 2)|^2} \\ &\times \mathrm{e}^{\pi\tilde{\nu}} M_{-\mathrm{i}\tilde{\nu}, \tilde{l}+\frac{1}{2}}(2\mathrm{i}\tilde{k}r_b) M_{i\tilde{\nu}, \tilde{l}+\frac{1}{2}}(-2\mathrm{i}\tilde{k}r_a). \qquad (42)\end{aligned}$$

Thus we have

$$\begin{aligned}\int_{Mc^2}^{\infty} \frac{\mathrm{d}E}{2\pi\hbar} \mathrm{disc}\, G_l(r_b, r_a; E > Mc^2) &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \frac{(\hbar c)^2 \tilde{k} \mathrm{d}\tilde{k}}{\sqrt{M^2c^4 + (\hbar c\tilde{k})^2}} \mathrm{disc} G_l(r_b, r_a; E > Mc^2) \\ &= -\frac{\mathrm{i}}{(r_b r_a)^{(D-1)/2}} \int_{-\infty}^{\infty} \mathrm{d}\tilde{k} \left(\frac{E}{Mc^2}\right) R_{\tilde{k}l}(r_b) R^*_{\tilde{k}l}(r_a). \qquad (43)\end{aligned}$$

From this, we obtain the continuous radial wavefunction of the D-dimensional relativistic Coulomb system

$$R_{\tilde{k}l}(r) = \sqrt{\frac{1}{2\pi}} \frac{1}{\left[1 + (\frac{c\hbar\tilde{k}}{Mc^2})^2\right]^{1/2}} \frac{|\Gamma(-\mathrm{i}\tilde{\nu} + \tilde{l} + 1)|}{(2\tilde{l} + 1)!} \mathrm{e}^{\pi\tilde{\nu}/2} M_{\mathrm{i}\tilde{\nu}, \tilde{l}+1/2}(-2\mathrm{i}\tilde{k}r) \qquad (44)$$

$$\begin{aligned}&= \sqrt{\frac{1}{2\pi}} \frac{1}{\left[1 + (\frac{c\hbar\tilde{k}}{Mc^2})^2\right]^{1/2}} \frac{|\Gamma(-\mathrm{i}\tilde{\nu} + \tilde{l} + 1)|}{(2\tilde{l} + 1)!} \mathrm{e}^{\pi\tilde{\nu}/2} \mathrm{e}^{\mathrm{i}\tilde{k}r} (-2\mathrm{i}\tilde{k}r)^{\tilde{l}+1} \\ &\quad \times M(-\mathrm{i}\tilde{\nu} + \tilde{l} + 1, 2\tilde{l} + 2; -2\mathrm{i}\tilde{k}r). \qquad (45)\end{aligned}$$

It is easy to check the result is in accordance with the non-relativistic wavefunction when we take the non-relativistic limit.

3. Concluding remarks

In this paper we have calculated the Green's function of the relativistic Coulomb system via sum over perturbation series. From the resulting amplitude, the energy levels and wavefunctions are given. Different from the conventional treatment in path integral using the spacetime and Kustaanheimo–Stiefel transformation techniques (e.g. [11, 13]), the method presented here just involves the computation of the expectation value of the moments Q^n $(Q = \int_{\tau_a}^{\tau_b} \mathrm{d}\tau \rho(\tau) V(\boldsymbol{x}(\tau)))$ over the measure

$$K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \tau_b - \tau_a) = \int D^D x \, \mathrm{e}^{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} \mathrm{d}\tau \left[\frac{M}{2\rho(\tau)} \boldsymbol{x}'^2(\tau) - \rho(\tau) \frac{V(\boldsymbol{x})^2}{2Mc^2}\right]}$$

and summing them in accordance with the Feynman–Kac formula [16]

$$\begin{aligned}G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) &= \frac{\mathrm{i}\hbar}{2Mc} \int_0^{\infty} \mathrm{d}S \int D\rho \Phi[\rho] \mathrm{e}^{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} \mathrm{d}\tau \rho(\tau) \mathcal{E}} \\ &\quad \times E\left[\exp\left\{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} \mathrm{d}\tau\, \rho(\tau) V(\boldsymbol{x}(\tau))\right\}\right] \\ &= \frac{\mathrm{i}\hbar}{2Mc} \int_0^{\infty} \mathrm{d}S \int D\rho \Phi[\rho] \mathrm{e}^{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} \mathrm{d}\tau \rho(\tau) \mathcal{E}} \sum_{n=1}^{\infty} \frac{(-\beta/\hbar)^n}{n!} \\ &\quad \times E\left[\left(\int_{\tau_a}^{\tau_b} \mathrm{d}\tau\, \rho(\tau) V(\boldsymbol{x}(\tau))\right)^n\right] \qquad (46)\end{aligned}$$

where the notation $E\,[\star]$ stands for the expectation value of the moment $\star$.

We hope that the procedure presented in this paper may help us to obtain the results of other interesting relativistic systems.

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